Re-examining the Cost-of-Living Index and the Biases of Price Indices

Implications for the U.S. CPI

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ABSTRACT

The U.S. CPI is based on the Laspeyres price index, an index type that has an upward "substitution bias." Thus, the CPI tends to overstate increases in the cost of living. To address this bias, the Advisory Commission to Study the Consumer Price Index recommended adopting for the CPI a "superlative" price index, e.g., the Fisher or Tornqvist indices. Under the assumption of *homothetic preferences*, superlative indices always have smaller substitution biases—hence, are closer to the "true" cost-of-living index (COLI)—than the Laspeyres index, but this assumption implies that: all income elasticities equal 1, the true COLI is independent of the utility level (standard of living), and expenditure shares are unaffected by changes in income. These implications are contradicted, however, by all known household budget studies. Therefore, superlative indices are not necessarily closer to the true COLI than the Laspeyres index except in the unrealistic case of homothetic preferences. Under more realistic non-homothetic preferences, expenditure shares vary with income and, thus, "income bias" is introduced into the superlative indices. This, in turn, could result in biases larger than the Laspeyres substitution bias in the CPI. The Commission did not, however, address this possibility. The Laspeyres index has a larger substitution bias but no income bias because it uses fixed expenditure shares. Under plausible conditions, by using a non-homothetic "almost ideal demand system" (AIDS) model, we carry out empirical simulations that show that the *combined* substitution and income biases of either the Fisher or the Tornqvist index could be either *positive* or *negative*—that is, a superlative index could differ even more from the true COLI than is the case for the present CPI. Thus, income adjustments resulting from a CPI based on a superlative index could exceed those from using the current CPI, when the combined bias is positive, or could fall below those warranted by the true COLI in the case of negative combined bias. Therefore, the Commission's recommendation for a superlative index CPI formula needs further examination. We propose a theoretically rigorous and practical procedure to determine a COLI free from substitution and income bias, using estimated ordinary demand functions without postulating a specific structure of consumer preferences.

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By Jesus C. Dumagan and Timothy D. Mount*

I. INTRODUCTION

The appointment of the Advisory Commission to Study the Consumer Price Index by the U.S. Senate Finance Committee was triggered by Alan Greenspan's remark (1995) before the joint Budget Committees of Congress that: "The official CPI may currently be overstating the increase in the true cost of living by perhaps 1/2 percent to 1-1/2 percent per year. ... If the annual inflation adjustments to indexed programs and taxes were reduced by 1 percentage point ... the annual level of the deficit will be lower by about $55 billion after five years." The Commission issued a final report (1996) on the problems of the CPI and recommendations to remedy its shortcomings.[1]

The overstatement of the true cost-of-living index (COLI) by the CPI has long been recognized (Stigler, 1961; Noe & von Furstenberg, 1972; Triplett, 1973). It reflects in part the positive substitution bias inherent in the Laspeyres price index. However, except for substitution bias and the income bias that was not addressed by the Commission, we do not analyze the other biases discussed in the Commission's final report.[2] Our major focus in this

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[1]The members of the Commission were Michael J. Boskin, Chairman (Professor of Economics, Stanford University), Ellen R. Dulberger (Program Director, IBM Global I/T Services Strategy & Economic Analysis), Robert J. Gordon (Professor of Economics, Northwestern University), Zvi Griliches (Professor of Economics, Harvard University), and Dale Jorgenson (Professor of Economics, Harvard University).

[2]In its final report, the Commission presented estimates, in percentage points per year, of the major “biases” in the CPI. *Substitution bias*—consisting of bias at the upper level of aggregation (0.15) and at the lower level of aggregation (0.25)—is due to the “fixed” basket assumption of the CPI and occurs because consumers tend to substitute less expensive for more expensive goods when relative prices change. *New products bias* (0.60, together with *quality change bias*) is introduced when

paper is on the Commission's recommendation that the CPI should move toward a COLI concept by adopting a "superlative" index formula, e.g., the Fisher or Tornqvist. Such superlative indices allow for changing market baskets in place of the fixed-weight Laspeyres formula now used in the CPI calculation. Our analysis draws on the relevant technical and practical aspects of economic theory, in the spirit of President Clinton's recent statement that the "true measure of inflation and the question of accurately adjusting federal cost-of-living adjustments ... should be determined based on broad-based agreement among top technical experts and done on a technical basis, and that it should not be done for budgetary or political reasons."[3]

The assumption of homothetic preferences underlies superlative indices. This assumption implies that: all income elasticities for consumer goods equal 1, the true COLI is independent of the utility level (standard of living), and expenditure shares are unaffected by changes in income. The assumption of homothetic preferences, however, "contradicts all known household budget studies, not to mention most of the time-series evidence of systematic change in expenditure patterns as total outlays increase" (Deaton & Muellbauer, 1980b, p. 144). Thus, changing the CPI from the Laspeyres to a superlative price index formula could introduce an "income bias" in the more realistic case of non-homothetic preferences. We address income bias in this paper to stress the importance of the warning that: "Users of superlative indexes should recognize that income effects can matter, especially for comparisons over long periods of time ... and avoid confusing these income effects with substitution effects from price changes" (Moulton, 1996, p. 165).

The magnitude of the combined substitution and income biases of either the Fisher or the Tornqvist indices could either be negative or positive. They could also be larger than the substitution bias of the Laspeyres index. Therefore, a CPI based on a superlative index may result in either undercompensation or overcompensation relative to the income adjustment obtained from the true COLI. In fact, overcompensation may exceed that from the use of the current CPI due to substitution bias. Numerical simulations under plausible conditions using a non-homothetic AIDS model (Deaton & Muellbauer, 1980a, 1980b) demonstrate these possibilities. We use the AIDS model because it has gained wide implementation in practice and it has a "true" COLI to which we can compare the Laspeyres, Paasche, Fisher, and Tornqvist price indices.

Estimates of substitution bias and income bias are conjectural when we do not know the actual demand functions of the goods in the CPI market basket. The current CPI framework uses Laspeyres formulations both at the lower-level of aggregation (*entry level item* in BLS terminology) and at the upper-level of aggregation (product categories or *strata*). The

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new goods are not included in the goods basket of the CPI or introduced with a long lag. *Quality change bias* arises when improvements in the quality of products (e.g., higher energy-efficiency or less need for repair) are measured inaccurately or not considered at all. *Outlet substitution bias* (0.10) arises when shifts to outlets with lower prices are not taken into account.

[3]From a statement to reporters by National Economic Council Chairman Gene Sperling on behalf of the President (The Bureau of National Affairs, February 10, 1997).

demand systems approach to the COLI is not feasible at the lower level because the necessary data are not collected. However, it is feasible at the upper level as shown by Braithwait (1980) who derived the COLI from a demand system with a specific utility function.[4] Vartia (1983) devised an algorithm to compute the COLI from ordinary demand functions *without* an explicit parametric form of the utility function. However, we use an improved alternative to Vartia's algorithm (Dumagan & Mount, 1995; forthcoming in *Economics Letters*, 1997) to derive from an ordinary demand system a COLI free from substitution and income biases. While implementing the demand systems approach to the COLI may impose transitional problems in the framework of timely production of the CPI, we propose that this approach be investigated as part of current BLS research initiatives (Abraham, 1997) to understand better the biases of the CPI and to move the CPI towards a COLI concept.

II. THE TRUE COST-OF-LIVING INDEX (COLI) AND PRICE INDICES

The true COLI is the *ratio* of the minimum expenditure at alternative prices to the minimum expenditure at base prices while keeping the standard of living unchanged.[5] To derive the COLI, consider a situation where prices and quantities are given by the vectors p and q. These vectors encompass n goods for which a consumer has a utility function $U(q)$. Thus,

$$(1) \qquad p = \{p_i\} \;\;,\;\; q = \{q_i\} \;\;,\;\; U = U(q) \;\; for \;\; i = 1,\, 2,\, ...,\, n\,.$$

In any situation, s, the consumer minimizes expenditure given the prices p^S to attain a predetermined utility level U^S. The consumer's minimum expenditure, $C(p^S, U^S)$, is the solution to the problem of determining the bundle q such that $U(q)$ is at least equal to the level of U^S. That is,

$$(2) \qquad C(p^S, U^S) \;=\; MIN_q \left[\sum_{i=1}^{n} p_i^S q_i \;:\; U(q) \;\geq\; U^S \right].$$

Suppose that prices change from the original price vector p^O to a terminal price vector p^T, while staying on the original indifference curve U^O. In this case, total minimum expenditure changes from $C(p^O, U^O)$ to $C(p^T, U^O)$. To express this change formally, let t be an auxiliary

[4]Braithwait (1980) employed a linear expenditure system (Klein & Rubin, 1948), which is restrictive because the goods can only be Hicksian substitutes (Phlips, 1983). According to this demand system, if you are buying more then two commodities, two of which are socks and shoes, for example, you cannot maintain the same level of utility when relative prices change by buying more socks when you buy more shoes. Also, Braithwait was constrained by the state of the art at that time when the demand systems approach to the COLI required a parametric specification of the utility function, which is not necessary anymore as shown by Vartia (1983).

[5]The conceptual basis of the U.S. CPI is the COLI from the theory of expenditure minimization (Gillingham, 1974; Pollak, 1989).

variable in the interval $0 \le t \le T$ such that $p(t)$ is a differentiable price curve connecting p^O to p^T. Continuity in prices of the expenditure function implies that

$$(3) \qquad C(p^T, U^O) = C(p^O, U^O) + \sum_{i=1}^{n} \int_O^T \frac{\partial C(p(t), U^O)}{\partial p_i(t)} \, dp_i(t)$$

$$= C(p^O, U^O) + \sum_{i=1}^{n} \int_O^T q_i^h(p(t), U^O) \, dp_i(t) \, ,$$

where $q_i^h(p(t), U^O)$ is a (Hicksian) compensated demand function obtained by invoking Shephard's lemma.

The true COLI, the Laspeyres price index, and the substitution bias of the latter index can be derived from a second-order Taylor series approximation to (3). For this purpose, let

$$(4) \qquad \Delta p_i = p_i^T - p_i^O \quad , \quad p^O = \{p_i^O\} \quad and \quad p^T = \{p_i^T\} \quad for \quad i = 1, 2, ..., n \, .$$

By duality between expenditure minimization and utility maximization,

$$(5) \qquad \frac{\partial C(p(t), U^O)}{\partial p_i} = q_i^h(p(t), U^O) = q_i(p(t), C(p(t), U^O)) \, ,$$

where $q_i(p(t), C(p(t), U^O))$ is the ordinary demand function. That is, when compensated income $C(p(t), U^O)$ is substituted into the ordinary demand functions, the quantities obtained are the compensated quantities.[6] Moreover, the compensated price effect is given by the Slutsky equation

$$(6) \qquad \frac{\partial^2 C(p,U)}{\partial p_i \partial p_j} = \frac{\partial q_i^h}{\partial p_j} = \frac{\partial q_i}{\partial p_j} + q_j \frac{\partial q_i}{\partial C} = S_{ij} \, ,$$

where S_{ij} is an element of the Hicks-Slutsky substitution matrix (see p. 6).

Using (4), (5) and (6), the second-order Taylor approximation to (3) is

$$(7) \qquad C(p^T, U^O) \approx C(p^O, U^O) + \sum_{i=1}^{n} q_i(p^O, C(p^O, U^O)) \Delta p_i$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} S_{ij} (p^O, C(p^O, U^O)) \, \Delta p_i \Delta p_j \, .$$

In (7), $q_i(p^O, C(p^O, U^O))$ is the expenditure-minimizing quantity of each good on U^O, given p^O. Let this be denoted by q_i^O, in short, and let these individual quantities be the elements of

[6]Compensated income is the income that maintains the same level of utility after a price change. It is the income that results after adjusting the original income or budget for the "income effect," following the Slutsky decomposition between the "substitution effect" and the "income effect" of relative price changes.

the quantity vector $q^O = \{ q_i^O \}$, $i = 1, 2, ..., n$. Also, let $p^j \bullet q^k$ be the dot product of the price vector p^j and the quantity vector q^k. Using these definitions and (4), the second term on the right-hand side of (7) becomes

$$(8) \qquad \sum_{i=1}^{n} q_i(p^O, C(p^O, U^O))\,\Delta p_i = \sum_{i=1}^{n} q_i^O p_i^T - \sum_{i=1}^{n} q_i^O p_i^O = p^T \bullet q^O - p^O \bullet q^O .$$

By duality, observed expenditures equal compensated income at the original prices. That is,

$$(9) \qquad C(p^O, U^O) = p^O \bullet q^O .$$

It follows from (7), (8) and (9) that

$$(10) \qquad \frac{C(p^T, U^O)}{C(p^O, U^O)} \approx \frac{p^T \bullet q^O}{p^O \bullet q^O} + \frac{1}{2\,(p^O \bullet q^O)} \sum_{i=1}^{n} \sum_{j=1}^{n} S_{ij}\,(p^O, C(p^O, U^O))\,\Delta p_i \Delta p_j .$$

In (10), the left-hand term is the COLI for the standard of living U^O, the first term on the right-hand side is the Laspeyres price index, and the last term is the second-order substitution term—except for the missing remainder of the Taylor approximation.[7]

Similarly, a second-order Taylor approximation to $C(p^O, U^T)$ starting from $C(p^T, U^T)$ yields the analogous expression to (10), which is

$$(11) \qquad \frac{C(p^O, U^T)}{C(p^T, U^T)} \approx \frac{p^O \bullet q^T}{p^T \bullet q^T} + \frac{1}{2\,(p^T \bullet q^T)} \sum_{i=1}^{n} \sum_{j=1}^{n} S_{ij}\,(p^T, C(p^T, U^T))\,\Delta p_i \Delta p_j .$$

The left-hand term is the inverse of the COLI, $C(p^T, U^T)/C(p^O, U^T)$, for the living standard U^T, while the first term on the right-hand side is the inverse of the Paasche price index, and the last is the second-order substitution term.

Substitution Bias of the Laspeyres and Paasche Price Indices

The last terms in (10) and (11) reflect substitution via the Hicks-Slutsky matrix of substitution effects, S_{ij}. These terms are expressed in a quadratic form that is non-positive *no matter what happens to prices*, due to the concavity in prices of the expenditure function implied by expenditure minimization (Varian, 1992). Non-positive includes zero, but if the term is non-zero, the value of this quadratic form is always negative—thus, a measure of "substitution bias"—if the approximation is truncated after the first-order and we are left with the Laspeyres or the Paasche price index.

There are, however, two situations when the above quadratic form is zero and, therefore, no substitution bias exists for either the Laspeyres or Paasche indices. One is when preferences are Leontief-type because, in this case, each element of the substitution matrix is zero, i.e.,

[7]The missing remainder term is an additional source of "substitution bias" on top of the second-order term if the approximation is truncated after the first-order.

it is a null matrix. The other is when prices change proportionately because, in this situation, the quadratic form is identically zero for proportional changes in prices, no matter the underlying preferences.[8] Except for these two instances, however, the quadratic form is negative and, therefore, the Laspeyres and Paasche indices have substitution bias. Hence, from (10),

$$(12) \qquad \frac{1}{2\,(p^{O} \bullet q^{O})} \sum_{i=1}^{n} \sum_{j=1}^{n} S_{ij}\,(p^{O}, C(p^{O}, U^{0}))\,\Delta p_i \Delta p_j \;\le\; 0$$

$$\textit{implies that} \quad \frac{C(p^{T}, U^{O})}{C(p^{O}, U^{O})} \;\le\; \frac{p^{T} \bullet q^{O}}{p^{O} \bullet q^{O}}\,.$$

This is the well-known result that the Laspeyres price index is the *upper bound to the true COLI for* U^{O} (Konüs, 1939; Deaton and Muellbauer, 1980b; Pollak, 1989). Similarly, from (11),

$$(13) \qquad \frac{1}{2\,(p^{T} \bullet q^{T})} \sum_{i=1}^{n} \sum_{j=1}^{n} S_{ij}\,(p^{T}, C(p^{T}, U^{T}))\,\Delta p_i \Delta p_j \;\le\; 0$$

$$\textit{implies that} \quad \frac{p^{T} \bullet q^{T}}{p^{O} \bullet q^{T}} \;\le\; \frac{C(p^{T}, U^{T})}{C(p^{O}, U^{T})}\,,$$

which shows that the Paasche price index is the *lower bound to the true COLI for* U^{T}. For similar reasons given earlier, the Paasche price index has no substitution bias—i.e., it is the exact COLI for U^{T}—under the limiting assumption that Leontief-type preferences obtain or, alternatively, given proportional changes in prices.

Except for the two situations above, the substitution terms—the quadratic forms in (12) and (13)—are non-positive no matter what happens to prices. These terms measure substitution between the components of the goods bundle along the indifference curve U^{O} or U^{T} as prices change from p^{O} to p^{T}. Thus, the formulas of the Laspeyres and Paasche price indices have "substitution bias" because these formulas do not include a measure of substitution effects.

Alternatives to the Laspeyres and Paasche Price Indices

Measuring the COLIs shown in (10) and (11) requires as much data as are required to estimate a complete demand system encompassing the goods and services in the COLI market basket, because the substitution terms cannot be known without this demand system. In view of this data requirement, the Laspeyres and Paasche price indices have practical appeal since

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[8] The expenditure function is homogeneous of degree one in prices. Because the compensated demand function is, by Shephard's lemma, the first derivative with respect to own-price of the expenditure function, the compensated demand function is zero-degree homogeneous in prices. Since the substitution effects are the first derivatives of the compensated demand functions with respect to prices, it follows from Euler's theorem on homogeneous functions that the above quadratic form is identically zero when prices change in the same proportion because the theorem's result follows by factoring out the constant of proportionality.

they require only *two* sets of price-quantity observations—p^O, q^O, p^T, and q^T. These indices also have theoretical appeal because, as discussed above, they can closely approximate or even equal the COLI under certain conditions. However, as only first-order approximations to the true COLIs, the Laspeyres and Paasche price indices are subject to substitution bias. Alternative indices have been sought that use the *same* two sets of observations above but have *lower* substitution bias or are *closer* to the true COLI than the Laspeyres and Paasche indices. Among the alternatives are "superlative" indices such as the Fisher and Tornqvist price indices. However, whether or not superlative indices are closer to the COLI depends on the structure of consumer preferences and, as we shall see, these indices also are subject to biases.[9]

To facilitate comparison, we will express these various index types in common terms. Because the Tornqvist price index (Deaton & Muellbauer, 1980b) is expressed in terms of price ratios and expenditure shares, let us rewrite the Laspeyres, Paasche, and Fisher indices in a similar manner. By definition of expenditure shares,

$$(14) \qquad w_i^O = \frac{q_i^O p_i^O}{\sum_{i=1}^{n} q_i^O p_i^O} \;, \quad w_i^T = \frac{q_i^T p_i^T}{\sum_{i=1}^{n} q_i^T p_i^T} \;, \quad \sum_{i=1}^{n} w_i^O = 1 \quad and \quad \sum_{i=1}^{n} w_i^T = 1 \,.$$

Using (14), the Laspeyres (I_p^L) and Paasche (I_p^P) price indices become

$$(15) \qquad I_p^L = \frac{p^T \bullet q^O}{p^O \bullet q^O} = \sum_{i=1}^{n} w_i^O \left(\frac{p_i^T}{p_i^O} \right) \quad and \quad I_p^P = \frac{p^T \bullet q^T}{p^O \bullet q^T} = \frac{1}{\sum_{i=1}^{n} w_i^T \left(\frac{p_i^O}{p_i^T} \right)} \,.$$

The Fisher price index (I_p^F) is the geometric mean of the Laspeyres and Paasche indices. That is,

$$(16) \qquad I_p^F = (I_p^L \, I_p^P)^{\frac{1}{2}} = EXP \left\{ \frac{1}{2} \ln \left(\sum_{i=1}^{n} w_i^O \left(\frac{p_i^T}{p_i^O} \right) \right) - \frac{1}{2} \ln \left(\sum_{i=1}^{n} w_i^T \left(\frac{p_i^O}{p_i^T} \right) \right) \right\} .$$

The Tornqvist price index (I_p^T) is

[9]Diewert (1976) defined a price index as "superlative" if it is exact for a (unit) expenditure function which is capable of providing a second-order differential approximation to an arbitrary twice continuously differentiable linearly homogeneous (unit) expenditure function. The Laspeyres and Paasche price indices are not superlative because they are only first-order Taylor approximations to the COLIs. However, being superlative is neither necessary nor sufficient for the absence of substitution bias, because this depends on consumer preferences. For example, the Laspeyres and Paasche indices are exact COLIs—have no substitution bias—under Leontief-type preferences. However, superlative indices have substitution bias *if the true consumer preferences are not the same as the theoretical preference structure that rationalizes the superlative index as the true COLI.*

$$(17) \qquad I_p^T = EXP\left\{\frac{1}{2}\sum_{i=1}^{n}(w_i^O + w_i^T)\ln\left(\frac{p_i^T}{p_i^O}\right)\right\}.$$

In the above indices, the expenditure shares or weights, w_i^O and w_i^T, are not necessarily from the same indifference curves, because the Laspeyres standard of living (U^O) and the Paasche standard of living (U^T) need not represent the same level of utility. Thus, it is problematic for these shares to be combined in the Fisher index in (16), or in the Tornqvist index in (17), because the resulting standard of living is neither U^O nor U^T.

The standard of living represented by the expenditure shares w_i^O and w_i^T is of no concern, however, under the unrealistic assumption of homothetic preferences. In this case, the utility function $U(q)$ can be expressed as $U(q) = f(m(q))$, where f is a strictly increasing function and m is homogeneous of degree one in the goods bundle q (Deaton & Muellbauer, 1980b; Cornes, 1992; Varian, 1992).[10] Hence, the level of utility U is proportional to q and, given the prices p, the expenditure function $C(p, U)$ is proportional to U. That is,

$$(18) \qquad C(p,U) = U g(p),$$

for some function $g(p)$ that is homogeneous of degree one and concave in p. By way of Shephard's lemma, it follows from (18) that the expenditure share is independent of U since

$$(19) \qquad w_i = \frac{q_i^h p_i}{C(p,U)} = \frac{\partial C(p,U)}{\partial p_i}\frac{p_i}{C(p,U)} = \frac{\partial \ln C(p,U)}{\partial \ln p_i} = \frac{\partial \ln g(p)}{\partial \ln p_i}.$$

Therefore, the Laspeyres, Paasche, Fisher, and Tornqvist price indices depend only on prices or are invariant with respect to the level of utility—thus, also invariant to the level of income—*if* preferences are homothetic. In this special case, it does not matter if the expenditure shares w_i^O and w_i^T are on the same indifference curve or not. That is, it does not matter if the expenditure shares are from compensated income or from ordinary income *if* preferences are homothetic.

It follows from (18) that the true COLI is independent of the level of utility because

$$(20) \qquad \frac{C(p^T,U^O)}{C(p^O,U^O)} = \frac{C(p^T,U^T)}{C(p^O,U^T)} = \frac{g(p^T)}{g(p^O)}.$$

Recall from (12) that the Laspeyres index is the upper bound to the true COLI for U^O and from (13) that the Paasche index is the lower bound to the true COLI for U^T. Therefore, (12), (13), (15) and (20) imply that

[10] If the utility function is homothetic, then the *ordinality* of utility allows this homothetic function to be expressed as a monotonic increasing transformation of a linearly homogeneous utility function.

$$(21) \qquad \sum_{i=1}^{n} w_i^{O}\left(\frac{p_i^{T}}{p_i^{O}}\right) \;\geq\; \frac{g(p^{T})}{g(p^{O})} \;\geq\; \frac{1}{\sum_{i=1}^{n} w_i^{T}\left(\frac{p_i^{O}}{p_i^{T}}\right)}\,.$$

It is only under homothetic preferences that the true COLI, $g(p^{T})/g(p^{O})$ in (20), is independent of the level of utility. Therefore, homotheticity is necessary and sufficient for the inequality in (21) to obtain—where the true COLI cannot be larger than the Laspeyres and cannot be smaller than the Paasche (Fisher & Shell, 1972). This is the *only* case where the Laspeyres index cannot be smaller than the Paasche index. Thus, the "limiting" properties of the Laspeyres and Paasche indices—as the upper and lower bounds, respectively, to the true COLI, $g(p^{T})/g(p^{O})$—require homotheticity and homothetic preferences provide a self-fulfilling rationalization of the Fisher and the Tornqvist indices as approximations to the true COLI. By definition of a geometric mean, the Fisher index must be closer to this true COLI than either the Laspeyres or the Paasche index. Diewert (1976, 1992) showed that superlative indices provide close approximations to any COLI *if* preferences are homogeneous.[11] In particular, the Tornqvist is the COLI for homogeneous "translog" preferences and the Fisher is the COLI for homogeneous "quadratic" preferences.[12]

Income Bias of the Fisher and Tornqvist Price Indices

The significant implication of the assumption of homothetic preferences for the computation of price indices is that it does not matter if the expenditure shares w_i^O and w_i^T are from compensated income or from ordinary income. Moreover, because the true COLI is in this case independent of the level of utility, it is of no concern if U^O and U^T are the same. However, as previously noted, the necessary implications of homothetic preferences contradict all known household budget studies and most time-series evidence of systematic change in expenditure patterns as total outlays increase. Therefore, it is virtually certain that preferences are *not* homothetic in most actual situations.

Non-homotheticity could be problematic for the Fisher and Tornqvist price indices because they use expenditure shares from both time periods, i.e., both w_i^O and w_i^T. For these indices to measure the COLI, $C(p^T, U^O)/C(p^O, U^O)$, the share w_i^O should be from $C(p^O, U^O)$ and w_i^T should be from $C(p^T, U^O)$. Computing w_i^O is not a problem because $C(p^O, U^O) = p^O \bullet q^O$ is the known base period income. In contrast, w_i^T poses a problem because $C(p^T, U^O)$ is compensated income, which is unobservable and unknown without knowledge, at least, of the

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[11]In general, a homogeneous function is homothetic but a function can be homothetic without being homogeneous. It follows that (18) is the result even if the original utility function is non-homogeneous provided that it is homothetic.

[12]Diewert also showed that the Tornqvist price index is the COLI for a non-homothetic translog expenditure function if (I) a particular cardinalization of the utility function and (ii) the geometric mean of the utilities in the reference and comparison periods—to serve as the base utility level—are chosen.

ordinary demand functions.[13] Therefore, in practice, w_i^T is computed from $p^T \bullet q^T$, an observed expenditure level or ordinary income. However, $C(p^T, U^O)$ and $p^T \bullet q^T$ will yield the same w_i^T for the same prices if and only if preferences are homothetic.[14] Therefore, the Fisher and Tornqvist price indices will each differ from the COLI, $C(p^T, U^O)/C(p^O, U^O)$, if preferences are non-homothetic and this difference is mostly "income bias" because these indices have small substitution biases. The Paasche suffers from income bias as well, because it too uses w_i^T. Only the Laspeyres is free from income bias because it uses only w_i^O, which is fixed by definition of base period income.

Therefore, there is no necessary relationship among the Laspeyres, Paasche, Fisher, and Tornqvist price indices under non-homothetic preferences, except that the Fisher is always between the Laspeyres and Paasche by construction. The income bias could make the Paasche, Fisher, and Tornqvist indices *all smaller* or *all larger* than the Laspeyres. Consider that the standard of living of the COLI bounded from above by the Laspeyres is the CPI's base period standard of living because the CPI is a Laspeyres-type index. Hence, if the CPI is changed to a Fisher or Tornqvist formula, the possibility of a positive income bias could further overstate the current CPI *overcompensation* above the true COLI. It is also possible for the income bias to be negative, resulting in *undercompensation* by a CPI that is below the true COLI.

III. EFFECTS OF PRICE INDEX BIASES ON INCOME ADJUSTMENTS AND THE STANDARD OF LIVING

The true COLI, $C(p^T, U^O)/C(p^O, U^O)$, is the implicit reference index in income adjustment desired to maintain the original standard of living U^O. This income adjustment is desired because nominal incomes do not necessarily keep pace with the true cost of living. If we let Y^O be the nominal income at the original prices p^O and Y^T be the nominal income at the current prices p^T, then

[13] Technically, compensated income is the value of the minimum expenditure function $C(p, U)$ for a specific set of prices in the vector p, given a fixed utility level U. It is unobservable because it is a function of utility which is unobservable. Compensated income is, however, measurable given the ordinary demand functions even without positing a specific utility function. This is shown in section VI of this paper.

[14] Since $p^T \bullet q^T$ is an *observed* expenditure level, utility maximization implies that the bundle q^T is given by the tangency point between the budget line for $p^T \bullet q^T$ and an unknown indifference curve. The compensated income $C(p^T, U^O)$ is defined by a budget line that is tangent to the indifference curve U^O. The budget lines for $p^T \bullet q^T$ and $C(p^T, U^O)$ are *parallel* because they represent the same prices p^T. It is only in the case of homothetic preferences that tangency points between indifference curves and parallel budget lines will lie on the same straight line ray from the origin. This geometric result necessarily implies that the expenditure share w_i^T of each good from $p^T \bullet q^T$ is equal to its share from $C(p^T, U^O)$.

(22) $$Y^O = C(p^O, U^O) = p^O \bullet q^O \quad and \quad Y^T = p^T \bullet q^T .$$

The *current* nominal income Y^T is not necessarily equal to the *desired* nominal income Y^{TO}. By definition, Y^{TO} equals the original income Y^O multiplied by the COLI. The result, in turn, equals the compensated income, $C(p^T, U^O)$, that is necessary to maintain the standard of living U^O. That is,

(23) $$Y^{TO} = Y^O \times COLI = Y^O \left[\frac{C(p^T, U^O)}{C(p^O, U^O)} \right] = C(p^T, U^O) .$$

However, if the COLI is replaced by the Laspeyres price index, the result Y_L^{TO} may exceed compensated income or

(24) $$Y_L^{TO} = Y^O \left[\frac{p^T \bullet q^O}{p^O \bullet q^O} \right] = p^T \bullet q^O \geq C(p^T, U^O) .$$

This implies the possibility of income *overcompensation* and an adjustment in the standard of living above U^O. Since the Laspeyres index is an upper bound to the COLI, this price index cannot result in undercompensation.

In the limiting case where preferences are assumed to be homothetic, the Paasche is the lower bound to the same COLI that is bounded from above by the Laspeyres. Therefore, undercompensation or an unintended deterioration of the standard of living below U^O is possible if the Paasche price index is used in place of the COLI in (23). The Paasche index cannot, however, result in overcompensation under homothetic preferences. Moreover, in the simplistic case of homothetic preferences, the Fisher and the Tornqvist indices lie between the Laspeyres upper bound and the Paasche lower bound. Thus, if the COLI is replaced by either the Fisher or the Tornqvist, exact compensation or the maintenance of U^O is possible.

From (22) and (23), an income adjustment is warranted if the *desired* nominal income Y^{TO} is different from the *current* nominal income Y^T. Suppose that

(25) $$\frac{C(p^T, U^O)}{C(p^O, U^O)} > \frac{Y^T}{Y^O} \quad or \quad Y^{TO} = Y^O \left[\frac{C(p^T, U^O)}{C(p^O, U^O)} \right] > Y^T .$$

In (25), it happens that the true COLI is larger than the ratio of income Y^T in the current period to income Y^O in the base period. In this situation, the consumer's standard of living would worsen below U^O unless Y^T is adjusted upwards to the desired level Y^{TO}.[15]

The situation in (25) presents problems in practice, given the possibility that the price index chosen in place of the unknown COLI for U^O could be below this COLI in one situation or

[15]As an aside, the difference between Y^{TO} and Y^T in (25) is the Hicksian *compensating variation* allowing for a change in the current level of nominal income Y^T from the base level Y^O (Boadway & Bruce, 1984). If there is no change, this compensating variation is the same as the original Hicksian definition (Hicks, 1956).

above it in another. While the Laspeyres index used for the CPI could be in error, theory implies that the error could only be overcompensation because the Laspeyres cannot be lower than the COLI for U^O no matter what is the structure of consumer preferences. In contrast, under the more realistic cases of non-homothetic preferences, the Paasche, Fisher, and Tornqvist indices could be lower or higher than the above COLI and they could also be *larger* than the Laspeyres. Hence, they could result in income adjustments of the wrong size or, worse, in a direction detrimental to the consumer. We illustrate these possibilities in the following numerical simulations.

IV. NUMERICAL SIMULATIONS OF ANALYTIC FINDINGS

It follows from the preceding analysis that in the current Laspeyres framework of the CPI only *overcompensation* is possible. In contrast, a shift to either the Fisher or Tornqvist index could result not only in a larger overcompensation but also in undercompensation. We demonstrate these possibilities with numerical simulations under plausible conditions, using a non-homothetic "almost ideal demand system" or AIDS model (Deaton & Muellbauer, 1980a, 1980b).

We chose the AIDS model because it has gained wide acceptance in empirical applications and because it allows for non-homothetic preferences. This model suits our illustrative purposes because it has explicit indirect utility and expenditure functions, so that exact compensated incomes and the values of the "true" COLI for U^O can be computed for given sets of prices. Hence, it is possible to compute the exact substitution bias of the above price indices, given U^O. When income changes, the non-homothetic AIDS model allows us to compute the combined substitution and income biases relative to the true COLI for U^O of each of the Laspeyres, Paasche, Fisher, and Tornqvist indices.

The AIDS indirect utility and expenditure functions are, respectively,

$$(26) \qquad U(p,C) = \left(\frac{C}{A}\right)^{\frac{1}{B}} \quad and \quad C(p,U) = A\,U^{B},$$

where p is the price vector and C is income. Price functions A and B are defined by

$$(27) \qquad \ln A = \alpha_0 + \sum_{i=1}^{n} \alpha_i \ln p_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_{ij} \ln p_i \ln p_j \quad and \quad \ln B = \beta_0 + \sum_{i=1}^{n} \beta_i \ln p_i .$$

Combining (26) and (27), and then using Roy's identity, the indirect utility function yields the ordinary demand function $q_i\,(p, C\,)$. From this, the expenditure share out of ordinary income of good i in the AIDS model is

$$(28) \qquad w_i = \frac{p_i\, q_i(p,C)}{C} = \alpha_i + \sum_{j=1}^{n} \gamma_{ij} \ln p_j + \beta_i \ln\left(\frac{C}{A}\right).$$

This model is non-homothetic if and only if β_i is non-zero.[16] To insure that the sum of expenditure shares equals one, that zero-degree homogeneity in income and prices of the ordinary demand functions obtains, and that there is symmetry of the compensated price effects, the following restrictions are imposed on the parameters:

$$(29) \qquad \sum_{i=1}^{n} \alpha_i = 1 \;, \quad \sum_{i=1}^{n} \beta_i = 0 \;, \quad \sum_{i=1}^{n} \gamma_{ij} = 0 \;, \quad \sum_{j=1}^{n} \gamma_{ij} = 0 \quad \textit{and} \quad \gamma_{ij} = \gamma_{ji}.$$

For two goods, the AIDS ordinary price and income elasticities from (28) are:

$$(30) \qquad E_{ii} = \frac{\partial q_i}{\partial p_i}\frac{p_i}{q_i} = -1 + \frac{1}{w_i}\left\{\gamma_{ii} - \beta_i\left[w_i - \beta_i \ln\left(\frac{C}{A}\right)\right]\right\},$$

$$(31) \qquad E_{ij} = \frac{\partial q_i}{\partial p_j}\frac{p_j}{q_i} = \frac{1}{w_i}\left\{\gamma_{ij} - \beta_i\left[w_j - \beta_j \ln\left(\frac{C}{A}\right)\right]\right\}, \textit{ and}$$

$$(32) \qquad E_{iC} = \frac{\partial q_i}{\partial C}\frac{C}{q_i} = 1 + \frac{\beta_i}{w_i}.$$

By substituting the above elasticities into the Slutsky equation, the compensated own-price effect for either good is

$$(33) \qquad S_{ii} = -\frac{C}{p_i^2}\left[(1 - w_i)\, w_i - \gamma_{ii} - \beta_i^2 \ln\left(\frac{C}{A}\right)\right].$$

Since there are only two goods, it is necessary and sufficient for expenditure minimization that (33) be non-positive for either good.[17]

For the purposes of the simulation, the following parameter values were used:

$$(34) \qquad \alpha_0 = -0.15 \;, \; \beta_0 = 0.25 \;, \; \alpha_1 = -0.45 \;, \; \beta_1 = 0.10 \textit{ and } \gamma_{11} = -0.20.$$

These values were chosen principally for two reasons. One is that they make the two-good AIDS model well-behaved in all the simulations—i.e., they yield for each good, negative own-price compensated price effects at each set of prices. The other is that these parameter values

[16] If each β_i is zero, the expenditure shares in (28) are independent of income C and the expenditure function $C(p, U)$ in (26) is proportional to utility, which are implied by homotheticity.

[17] This condition is necessary but not sufficient if there are more than two goods. In the latter case, the necessary and sufficient condition for expenditure minimization is that the Hicks-Slutsky substitution matrix or the matrix of compensated price effects is symmetric and negative semi-definite (see equation (46), p. 28).

yield plausible expenditure shares, own-price and cross-price elasticities, and income elasticities, in terms of both the signs and ranges of values (see Table 4, p. 21).

The values of the other parameters, α_2, β_2, γ_{12}, γ_{21}, and γ_{22}, are obtained from (34) subject to the restrictions in (29). The data for the original prices and utility are

$$(35) \qquad \left\{p_1^{\,O},\ p_2^{\,O}\right\} \ = \ \{1.0000,\ 2.0000\} \quad \textit{and} \quad U^{\,O} \ = \ 100\,.$$

Given (34) and (35), the minimum expenditure solved by the model is $C^{\,O}$ = 557.9042. Since this minimum expenditure is for the original set of prices, it is used as the original income level for analytical purposes. Starting with the prices in (35), we generate the prices shown in Tables 1, 2, and 3 such that our computed Laspeyres price index grows at the same rate as the actual rates of change during 1983–85 of the official all-item CPI for urban consumers (CPI-U).[18] Compensated income is obtained by solving $C(p, U)$ from (26), while holding utility fixed at the level $U^O = U^T$ for T = 0, 1, ..., 12. The COLI for the prices in each step is, by definition, the ratio of compensated income for these prices to the original income $C^{\,O}$ = 557.9042, which is also the compensated income at the original prices. To compute the price indices, we determine the expenditure shares w_i^O and w_i^T out of compensated income at each set of prices T and, together with the prices, substitute them into the formulas in (15), (16), and (17), by letting T = 0, 1, ..., 12, in the tables. Notice that the Laspeyres index uses only w_i^O at T = 0 and the Paasche index uses only w_i^T for each T = 1, 2, ..., 12. In contrast, both the Fisher and the Tornqvist indices use w_i^O and w_i^T together as defined above. In Table 1, we calculate all the price indices using the expenditure shares w_i^O and w_i^T from the compensated income for each price set, given the fixed level of utility $U^{\,O}$ = 100 in (35). By definition of compensated income, we are moving along the same indifference curve as prices change. In this case, for each set of prices, subtracting the value of the true COLI from the value of a specific price index yields the "pure" substitution bias of the specific index.

As expected from theory, the Laspeyres index has a positive substitution bias because it is the upper bound and the substitution bias of the Paasche index is negative because it is the lower bound to the true COLI. Although the Laspeyres and Paasche indices do not necessarily bound the same true COLI under non-homothetic preferences, we forced them to bound the same true COLI in this non-homothetic case by keeping utility the same for both indices for the purposes of the simulation.

Notice that the Fisher and the Tornqvist indices also have positive substitution biases. The latter are, however, very much smaller than the Laspeyres bias. The Tornqvist index has the smallest bias. For the Laspeyres and Paasche indices, the substitution biases in Table 1 are

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[18]In the tables, the price of good 1 starts at 1 and increases by 0.05 at each step. Given this, the price of good 2, which starts at 2, is increased such that our computed Laspeyres index tracks the CPI-U during 1983–95. We let our Laspeyres price index equal 100 for the first set of prices in the tables. Implicitly, this corresponds to the year 1983 since CPI-U = 100 for 1982–84. During 1983–95, the average annual rate of change of the CPI-U was 3.6 percent, with the lowest rate of 1.9 percent in 1985–86 and the highest rate of 5.4 percent in 1989–90 (U.S. Department of Commerce, 1996).

equal to the values of the second-order terms in equations (10) and (11) plus the missing remainder terms. For the Fisher index, the bias is some exact combination of the Laspeyres

Table 1
The True COLI and "Pure" Substitution Bias of Price Indices in the AIDS Model
(Using Expenditure Shares From Compensated Income)

T	Price of Good 1	Price of Good 2	True COLI for $U^O = U^T$ (Index)	"Pure" Substitution Bias: Laspeyres Price Index	Paasche Price Index	Fisher Price Index	Tornqvist Price Index
0	1.0000	2.0000	100.0000	0.0000	0.0000	0.0000	0.00000
1	1.0500	2.0820	104.3150	0.0013	- 0.0013	0.0000	0.00000
2	1.1000	2.1500	108.0915	0.0094	- 0.0093	0.0000	0.00000
3	1.1500	2.1730	110.1177	0.0585	- 0.0570	0.0007	0.00001
4	1.2000	2.2450	114.0289	0.0837	- 0.0813	0.0012	0.00002
5	1.2500	2.3375	118.7394	0.0884	- 0.0858	0.0013	0.00002
6	1.3000	2.4575	124.5224	0.0651	- 0.0635	0.0008	0.00001
7	1.3500	2.6040	131.3254	0.0283	- 0.0278	0.0002	0.00000
8	1.4000	2.7170	136.8271	0.0203	- 0.0201	0.0001	0.00000
9	1.4500	2.7950	140.9803	0.0315	- 0.0310	0.0002	0.00000
10	1.5000	2.8750	145.2089	0.0432	- 0.0424	0.0004	0.00001
11	1.5500	2.9440	149.0094	0.0654	- 0.0639	0.0007	0.00001
12	1.6000	3.0240	153.2343	0.0807	- 0.0787	0.0010	0.00001

and Paasche biases, derivable from the geometric mean definition of the Fisher index. For the Tornqvist index, however, it is not clear how its substitution bias is related algebraically to the biases of the Laspeyres and Paasche indices because there is no explicit relationship between these three indices. Table 1, however, misrepresents actual practice because it assumes that we know the "true" compensated income in each price situation. If we did, the true COLI is known and price indices are needless. Thus, Table 1 defeats the purpose of price indices—to use them in normal situations when we do not know compensated income. Nevertheless, because this table measures only "pure" substitution bias, it provides a point of departure to see what happens in the non-homothetic AIDS model when the expenditure shares used by the indices are *not* from compensated income. We will now accommodate actual practice by using expenditure shares from ordinary income and by letting ordinary income change along with prices.

In Table 2, the only change in the simulation from Table 1 is to let the income level grow at the actual annual rate of change of per capita personal consumption expenditures (PCE) during 1983–95.[19] However, the true COLI remains the same as in Table 1, given the same utility level U^O and the same price series. The reason is that compensated income, which determines this true COLI, is determined only by prices when the *base level of utility is fixed* by the design of the simulation. This is necessary in order to have a fixed standard of living (U^O) as an analytical reference point for judging the welfare implications, or determining the biases of the other indices relative to the true COLI. Also, the Laspeyres index remains the same because it uses fixed expenditure shares from the base level of income. Thus, even when income grows, the standard of living of the Laspeyres is exactly the same as the fixed standard of living of the true COLI.

It is interesting to note in this "realistic" simulation that by letting income change exactly according to the actual rate of change of per capita PCE, the biases of the Paasche, Fisher, and Tornqvist indices all become positive. Subtracting the "pure" substitution biases from Table 1 gives the "income bias" relative to the true COLI for $U^{O.}$. Table 2 shows that the positive combined biases, which are almost entirely income biases, of each of the Fisher and Tornqvist indices are larger in most cases (7 out of 13), than the positive substitution bias of the Laspeyres index. In these seven cases, the superlative indices result in *overcompensation above that of the Laspeyres*. This is precisely what might happen if the CPI is changed from the Laspeyres formula to a "superlative" index formula.

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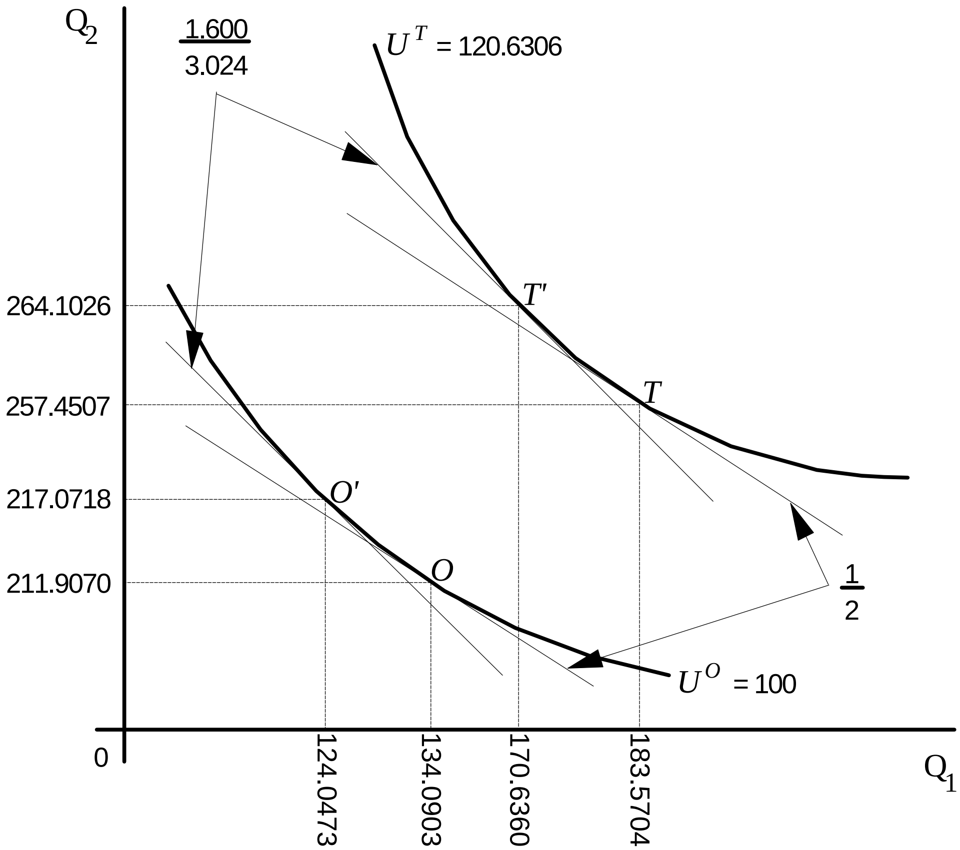
[19]During this period, the average annual rate of change of per capita PCE was 5.6 percent (U.S. Department of Commerce, 1996). The lowest annual rate was 2.4 percent in 1990–91 and the highest was 8.2 percent in 1983–84.

Table 2
The True COLI and Substitution and Income Biases of Price Indices in the AIDS Model
(Using Expenditure Shares From Ordinary Income Growing at the *Actual* Annual Rate of Change of Per Capita Personal Consumption Expenditures)

T	Price of Good 1	Price of Good 2	True COLI for U^O (Index)	Combined "Pure" Substitution and Income Bias: Laspeyres Price Index	Paasche Price Index	Fisher Price Index	Tornqvist Price Index
0	1.0000	2.0000	100.0000	0.0000	0.0000	0.0000	0.0000
1	1.0500	2.0820	104.3150	0.0013	0.0020	0.0016	0.0016
2	1.1000	2.1500	108.0915	0.0094	0.0090	0.0092	0.0092
3	1.1500	2.1730	110.1177	0.0585	0.0129	0.0357	0.0355
4	1.2000	2.2450	114.0289	0.0837	0.0212	0.0525	0.0522
5	1.2500	2.3375	118.7394	0.0884	0.0449	0.0666	0.0666
6	1.3000	2.4575	124.5224	0.0651	0.0612	0.0632	0.0633
7	1.3500	2.6040	131.3254	0.0283	0.0582	0.0432	0.0434
8	1.4000	2.7170	136.8271	0.0203	0.0475	0.0339	0.0341
9	1.4500	2.7950	140.9803	0.0315	0.0640	0.0478	0.0480
10	1.5000	2.8750	145.2089	0.0432	0.0788	0.0610	0.0613
11	1.5500	2.9440	149.0094	0.0654	0.1003	0.0828	0.0832
12	1.6000	3.0240	153.2343	0.0807	0.1144	0.0975	0.0980

Figure 1 illustrates the original price situation for $T = 0$ and for the terminal price situation $T = 12$ from Table 2. The original prices are 1 for good 1 and 2 for good 2. The expenditure-minimizing quantities at the original prices are 134.0903 for good 1 and 211.9070 for good 2, corresponding to the tangency point O on the indifference curve $U^O = 100$. The prices of both goods increase, from 1 to 1.600 for good 1 and from 2 to 3.024 for good 2. By definition, the Laspeyres price index is the ratio of the value of the original goods bundle at the new prices, (1.600)(134.0903) + (3.024)(211.9070) = 855.3512, to the value of the original bundle at the original prices, (1)(134.0903) + (2)(211.9070) = 557.9043. Therefore, the Laspeyres price index is (855.3512/557.9043)(100) = 153.3150 percent. The Laspeyres is the upper bound to the true COLI for U^O. This COLI is the ratio of the value of the goods bundle at the tangency point O', (1.600)(124.0473) + (3.024)(217.0718) = 854.9008 to the value of the goods bundle at the original tangency point O, which is 557.9043 as given earlier. Therefore, the true COLI for U^O is (854.9008/557.9043)(100) = 153.2343 percent,

Figure 1
The True COLIs and the Laspeyres and Paasche Price Indices



which is lower than the value of the Laspeyres price index. The difference of the Laspeyres from the true COLI is the substitution bias, 153.3150 - 153.2343 = 0.0807 for T = 12 in Tables 1 and 2. By the design of the simulation, given the step-by-step increase in the price of good 1 from 1 to 1.6, the price of good 2 was made to rise at each step such that the computed values of the Laspeyres index grew at the same rate as the actual rate of increase of the CPI-U during 1983–95. For T = 0, the Laspeyres equals 100 and this corresponds to CPI-U = 99.6 for 1983. For T = 12, the Laspeyres equals 153.3150 and this corresponds to CPI-U = 152.4 for 1995. Thus, the Laspeyres index increased from the original to the terminal prices at the same 53 percent rate as did the CPI-U from 1983 to 1995.

The original level of income or expenditure at point O is 557.9043. By design of the simulation, this value was solved by the model given the original prices and given that the original level of utility was set arbitrarily to U^O = 100. As the prices of the two goods increase, the original level of income or expenditure also increases to 1,071.6638, so that the consumer moves from tangency point O on U^O to the tangency point T' on U^T. This new expenditure level was not, however, chosen arbitrarily. By letting 557.9043 correspond to the level of per capita PCE in 1983, which was $9,744, the new expenditure level 1,071.6638 was chosen to correspond to the per capita PCE level in 1995, or $18,717. The reason is that in Table 2 we let income or expenditure grow from one price step to the next at the actual annual rate of increase of per capita PCE during 1983–95. Thus, 18,717/9,744 is equal to 1,071.6638/557.9043.

Given the terminal income of 1,071.6638, the maximum utility attained is U^T = 120.6306 at the terminal prices 1.600 for good 1 and 3.024 for good 2. In this case, the Paasche price index, by definition, is the ratio of the value of the terminal bundle at the terminal prices, (1.600)(170.6360) + (3.024)(264.1026) = 1,071.6638, at point T', to the value of the terminal bundle at the original prices, (1)(170.6360) + (2)(264.1026) = 698.8412. Therefore, the Paasche price index is (1,071.6638/698.8412)(100) = 153.3487 percent.

The Paasche is the lower bound to the true COLI for U^T. This COLI is the ratio of the value of the goods bundle at the tangency point T', which was previously calculated as 1,071.6638, to the value of the goods bundle at the tangency point T, (1)(183.5704) + (2)(257.4507) = 698.4718. Therefore, the true COLI for U^T is (1,071.6638/698.4718)(100) = 153.4298 percent, which is higher than the value of the Paasche price index.

The substitution bias of a price index is its difference from the corresponding true COLI, i.e., the difference of the Laspeyres from the COLI for U^O or the difference of the Paasche from the COLI for U^T. However, in the context of the current CPI controversy, the reference COLI is the true COLI for U^O because this corresponds to the Laspeyres price index that is the basis of the CPI formula. If preferences are homothetic, it would not matter if U^O and U^T are different. Under non-homothetic preferences, however, the difference matters, especially when income increases with prices because income bias arises in this case.

In Table 2, the difference of the Paasche price index from the COLI for U^O is the substitution bias from Table 1 combined with the income bias. At the prices for T = 12, the combined bias is 153.3487 - 153.2343 = 0.1144. In Table 1, the substitution bias at these prices of the Paasche index is 153.1556 - 153.2343 = -0.0787. Thus, the increase of the Paasche from 153.1556 for U^O to 153.3487 for U^T is the income bias, 0.1931, which more than offsets the negative substitution bias of -0.0787 to yield a combined positive bias of 0.1144 for T = 12 in Table 2.

In the above non-homothetic example, we showed that the Laspeyres price index (153.3150) is still the upper bound to the COLI for U^O (153.2343) and the Paasche price index (153.3487) is still the lower bound to the COLI for U^T (153.4298). Although non-homotheticity alone is necessary but not sufficient for the Paasche index to be larger than the Laspeyres, this happens to be true in this case for some prices. Accordingly, being the geometric mean of the Laspeyres and Paasche, the Fisher is larger than the Laspeyres. At the terminal prices, this is shown by the result that the combined substitution and income bias of the Fisher, 0.0975, is larger than the substitution bias of the Laspeyres, 0.0807. It also happens that the Tornqvist price index is larger because its combined bias of 0.0980 is larger than the Laspeyres substitution bias. In general, under non-homothetic preferences, Table 2 shows that the Paasche being larger than the Laspeyres is necessary and sufficient for the Fisher to be larger than the Laspeyres, by definition of the geometric mean. The Tornqvist index can also be larger than the Laspeyres index, as it is in some cases in Table 2.

In Table 3, everything else is the same as in Table 2, except that we now let the base year nominal income grow annually at a fixed rate of 2.4 percent, which was the lowest annual rate of change (1990–91) of per capita PCE during 1983–95. As shown, the income biases of the Paasche, Fisher, and Tornqvist indices are all negative. This implies that if the CPI is changed to a superlative index, there is a possibility of *undercompensation below the true COLI*, a rather perverse result considering that in this simulation real incomes are in fact

Table 3
The True COLI and Substitution and Income Biases of Price Indices in the AIDS Model
(Using Expenditure Shares From Ordinary Income Growing at the *Lowest* Annual Rate of Change of Per Capita Personal Consumption Expenditures)

T	Price of Good 1	Price of Good 2	True COLI for U^O (Index)	Combined "Pure" Substitution and Income Bias: Laspeyres Price Index	Paasche Price Index	Fisher Price Index	Tornqvist Price Index
0	1.0000	2.0000	100.0000	0.0000	0.0000	0.0000	0.0000
1	1.0500	2.0820	104.3150	0.0013	- 0.0029	- 0.0008	- 0.0008
2	1.1000	2.1500	108.0915	0.0094	- 0.0167	- 0.0037	- 0.0037
3	1.1500	2.1730	110.1177	0.0585	- 0.0722	- 0.0069	- 0.0077
4	1.2000	2.2450	114.0289	0.0837	- 0.1076	- 0.0121	- 0.0135
5	1.2500	2.3375	118.7394	0.0884	- 0.1265	- 0.0191	- 0.0208
6	1.3000	2.4575	124.5224	0.0651	- 0.1158	- 0.0254	- 0.0266
7	1.3500	2.6040	131.3254	0.0283	- 0.0772	- 0.0245	- 0.0250
8	1.4000	2.7170	136.8271	0.0203	- 0.0699	- 0.0248	- 0.0251
9	1.4500	2.7950	140.9803	0.0315	- 0.0969	- 0.0327	- 0.0333
10	1.5000	2.8750	145.2089	0.0432	- 0.1240	- 0.0404	- 0.0413
11	1.5500	2.9440	149.0094	0.0654	- 0.1666	- 0.0507	- 0.0521
12	1.6000	3.0240	153.2343	0.0807	- 0.1978	- 0.0586	- 0.0605

falling because the 2.4 percent rate of increase in nominal income is slower than the 3.6 percent average annual rate of change of the CPI-U during 1983–1995.

Tables 2 and 3 show that the income bias of the superlative indices could be positive or negative and, if positive, could be larger than the Laspeyres bias (Table 2). If the base year nominal income grows faster than in Table 2, the positive income biases get even larger. This implies that if the CPI is replaced with a superlative formula, the result could be *overcompensation above the current CPI indexation*, also a perverse result because this could happen even though real incomes are in fact rising.

Table 4 gives us an idea of the economic attributes of the goods in terms of the price and income elasticities for the simulation in Table 2, when the Laspeyres price index and the level of income track, respectively, the actual rates of change of CPI-U and per capita PCE during 1983–95. Between the two goods, good 1 is the more responsive with respect to both

Table 4
Expenditure Shares and Price and Income Elasticities in the AIDS Model
(Ordinary Income Growing at the *Actual* Annual Rate of Change of Per Capita Personal Consumption Expenditures in Table 2)

T	Good 1				Good 2			
	W_1	E_{11}	E_{12}	E_{1Y}	W_2	E_{21}	E_{22}	E_{2Y}
0	0.2403	-1.7026	0.2865	1.4161	0.7597	0.2223	-1.0907	0.8684
1	0.2428	-1.6949	0.2830	1.4119	0.7572	0.2228	-1.0907	0.8679
2	0.2444	-1.6890	0.2799	1.4091	0.7556	0.2229	-1.0905	0.8677
3	0.2435	-1.6889	0.2782	1.4107	0.7565	0.2217	-1.0895	0.8678
4	0.2444	-1.6855	0.2763	1.4092	0.7556	0.2217	-1.0894	0.8677
5	0.2473	-1.6774	0.2730	1.4044	0.7527	0.2226	-1.0897	0.8671
6	0.2502	-1.6703	0.2707	1.3996	0.7498	0.2237	-1.0903	0.8666
7	0.2534	-1.6636	0.2689	1.3947	0.7466	0.2252	-1.0913	0.8661
8	0.2525	-1.6663	0.2703	1.3960	0.7475	0.2251	-1.0913	0.8662
9	0.2535	-1.6633	0.2688	1.3945	0.7465	0.2252	-1.0913	0.8660
10	0.2540	-1.6614	0.2677	1.3937	0.7460	0.2252	-1.0912	0.8659
11	0.2545	-1.6594	0.2665	1.3929	0.7455	0.2251	-1.0910	0.8659
12	0.2548	-1.6584	0.2659	1.3925	0.7452	0.2251	-1.0909	0.8658

own-price and income. This is consistent with the fact that good 1 has the smaller expenditure share. In this case, good 1 is the "luxury" good and good 2 is a "necessity." Thus, good 2 has the larger expenditure share and lesser responsiveness to changes in prices and income. The positive cross-price elasticities imply that these goods are (gross) substitutes.

One remarkable result in Table 4 is the *stability* of the expenditure shares and the price and income elasticities. The absolute values of these elasticities may, however, seem large, but this is not surprising in a two-good model because these ordinary price and income elasticities are constrained to sum to zero for each combination of prices and income, as required by zero-degree homogeneity.[20] Given the realism of the simulation—by replication of the actual changes in the CPI-U and of per capita PCE—the stability of the expenditure shares and the price and income elasticities are significant because they imply that the perverse results for the superlative indices we noted in Tables 2 and 3 could happen under plausible conditions as well.

V. IMPLICATIONS FOR THE U.S. CPI

In the CPI framework (U.S. Department of Labor, 1987, 1992), the Laspeyres price index formula has been the basis for aggregation across strata of items (upper level aggregation). In the 1978 revision, the Laspeyres formula was adopted for calculation of the basic components under the entry-level method of sampling (lower level aggregation). Thus, there are two sources of substitution bias, at the upper and lower levels of aggregation. In this framework, while earlier studies concluded that the CPI substitution bias is relatively small quantitatively, Moulton (1996) implied that this issue is not yet settled.[21] He noted that existing studies of substitution bias (Manser & McDonald, 1988; Aizcorbe & Jackman, 1993) have been limited to examining substitution at the upper level between product categories.[22] That is, using Moulton's example, they were looking at substitution between canned soup and frozen meals, but not at substitution between types of frozen meals. In any case, these earlier studies on the substitution bias of the CPI computed the bias using the superlative price indices in place of the true COLI, as if these indices have no bias. Thus, the findings cannot be accepted uncritically because of the implicit and unrealistic assumption of homotheticity.

However, Moulton (1993) showed that lower-level substitution effects within product categories or strata are in some cases larger than the upper-level substitution effects between categories. Although his study was limited to data during the June 1992 to June 1993 period,

[20]The absolute sizes of ordinary price and income elasticities tend to be larger, the fewer the number of goods, in order to satisfy the above zero-sum constraint.

[21]Wynne & Sigalla (1994) provide a comprehensive survey of past and current studies on similar CPI biases identified by the Commission. We do not, however, share their contrary view that “the issue of substitution bias is the closest thing to being settled” (p. 17).

[22]Manser & McDonald used *Personal Consumption Expenditure* data covering the period 1959–85, whereas Aizcorbie & Jackman used data from the *Consumer Expenditure Survey* for the period 1982–91.

similar indications of important lower-level substitution effects were shown in studies by Reinsdorf (1996) and Bradley (1996) using supermarket scanner data. At the lower level, there are existing studies (Reisndorf, 1993a, 1993b; Moulton, 1993) indicating significant substitution bias by way of a geometric mean formula for the price relatives in place of the Laspeyres formulation.

The size of CPI substitution bias is logically separate, however, from the issue of the consistency of the CPI with the COLI concept of a fixed reference standard of living. As illustrated in Tables 1, 2, and 3, by using the *same* expenditure share weights (w_i^O, $I = 1, 2, ..., n$, defined on U^O for all prices, p_i^S, $s = 0, 1, ..., T$), the Laspeyres necessarily keeps U^O as the reference level of utility. Therefore, being a Laspeyres-type index, the CPI is consistent with the COLI concept of having a fixed standard of living, U^O, in the above example. This is not necessarily the case, however, with the other indices.

Both the Fisher and the Tornqvist use the expenditure shares w_i^O as weights for the prices p_i^O and the shares w_i^S for prices p_i^S, $s = 0, 1, 2, ..., T$. If $U^O = U^S$ for all s, like in Table 1, then we do not have the problem of a varying standard of living that complicates determination of the direction of welfare change given an actual income adjustment. We *do* have this problem in Tables 2 and 3, however, because $U^O \neq U^S$ and U^S is different for each situation s. Because the reference standard of living lies between the levels of utility U^O and U^S, this reference is not fixed.[23] Therefore, in the event of actual income adjustment, we have to know U^S to make a welfare judgement relative to the reference standard of living.

Therefore, contrary to the Commission's recommendation, a change of the CPI from the Laspeyres to a "superlative" index formulation very likely will produce a CPI divorced from the COLI concept; and, worse, a CPI that will make it difficult, if not impossible, for us to know the welfare effects of actual income adjustments. The reason from theory is that only overcompensation relative to the true COLI for U^O is possible from the positive substitution bias of the Laspeyres. In contrast, under non-homothetic preferences, the negative or positive income bias of the superlative indices could lead either to *undercompensation or overcompensation*, but there is no basis in theory to know which possibility holds in an actual setting. However, retaining the Laspeyres as the basis of the CPI has a downside, as shown by the fact that in going down the price steps in the tables, the substitution bias of the Laspeyres gets larger. This suggests that, in practice, the CPI "market basket" needs to be updated to alleviate the problem of an enlarging substitution bias over time caused by a market basket with fixed components.

[23]Deaton & Muellbauer (1980b) noted that Diewert (1976) showed that if the logarithm of the expenditure function is a quadratic form in the logarithms of prices and utility, then the Tornqvist index is the true COLI for (p^O, p^S, U^*) where the reference level of utility U^* is the geometric mean of U^O and U^S. However, Deaton & Muellbauer further noted that, without knowing the parameters of the expenditure function, “we lack more specific information about the reference indifference curve (such as what budget level and price vector corresponds to it), and the result is of no help in constructing a constant utility cost-of-living index series with more than two elements.”

VI. COMPUTING THE COLI FROM ORDINARY DEMAND FUNCTIONS

Without knowing the demand functions for the goods in the CPI market basket, estimates of substitution and income biases are conjectural. In the current CPI framework, the demand systems approach to the COLI is not feasible at the lower level of aggregation because the necessary data are not collected. However, it is feasible at the upper level because it has been done before (Braithwait, 1980).[24]

There are two ways to calculate the COLI in the demand systems approach. One is to utilize a demand system with an explicit parametric form of the indirect utility or expenditure function like the linear expenditure system (Klein & Rubin, 1948), translog (Christensen, Jorgenson, and Lau, 1975; Christensen and Caves, 1980) or the AIDS (Deaton and Muellbauer, 1980a and 1980b). In this case, *exact* computation of the COLI can be done because the demand system parameters are exactly the parameters of the utility and expenditure functions and this exact method is what we used to obtain the true COLI in the AIDS model.

The other way, to *approximate* the COLI from ordinary demand functions without an explicit utility or expenditure function, was demonstrated by Vartia (1983).[25] Vartia's algorithm marked a major milestone because until then computing the COLI required an explicit parametric form of the utility or expenditure functions. We have developed the REversible Second-ORder Taylor (RESORT) algorithm as an alternative to Vartia's algorithm (Dumagan & Mount, 1995; forthcoming in *Economics Letters*, 1997). RESORT utilizes the Slutsky equation to obtain from the ordinary demand functions the substitution matrix embodied in the second-order terms, and checks this matrix for symmetry and negative semi-definiteness to determine the consistency of the computed COLI with the theory of expenditure minimization. RESORT yields Vartia's algorithm as a special case when the second-order terms of RESORT are ignored. Thus, RESORT is an improvement over Vartia's algorithm because the latter does not have a built-in procedure to check for the theoretical validity of the computed COLI.

[24]Braithwait, who was then at the U.S. Bureau of Labor Statistics, computed the COLI and the bias of the Laspeyres price index by estimating a linear expenditure system (LES) encompassing ten commodity subgroups and six main groups. The price and quantity data are annual time-series data during 1948–73 from *Personal Consumption Expenditures in the United States*, published by the U.S. Department of Commerce.

LES was originally introduced by Klein & Rubin (1948) in an attempt to estimate a true COLI. It had been extensively applied to the estimation of COLIs in a number of European countries, the US and Japan over twenty years before Braithwait's application to US data. A description of LES and some of its applications are presented in Phlips (1983).

[25]For an empirical application of Vartia's algorithm, see Porter-Hudak & Hayes (1991).

The RESORT Algorithm for Computing the COLI

To provide a self-contained description here of our RESORT algorithm, we begin by repeating a few of the earlier mathematical steps in section II. To simplify notation, let $C^O = C(p^O, U^O)$ and $C^T = C(p^T, U^O)$. Borrowing from section II, let t be an auxiliary variable in the interval $0 \le t \le T$ such that $p(t)$ is a differentiable price curve connecting p^O to p^T. Continuity in prices of the expenditure function implies that

$$(36) \qquad C^T = C^O + \sum_{i=1}^{n} \int_O^T \frac{\partial C(p(t), U^O)}{\partial p_i(t)} \, dp_i(t)$$

$$= C^O + \sum_{i=1}^{n} \int_O^T q_i^h(p(t), U^O) \, dp_i(t) \,,$$

where $q_i^h(p(t), U^O)$ is a (Hicksian) compensated demand function by Shephard's lemma. While (36) computes $C^T = C(p^T, U^O)$, starting from $C^O = C(p^O, U^O)$ as prices change from and p^O to p^T, the procedure applies to measuring $C(p^O, U^T)$ from $C(p^T, U^T)$ as p^T change back to p^O.

Break up the price change from $p^O = \{ p_i^O \}$ to $p^T = \{ p_i^T \}$ into price steps, at each s = 0, 1, ..., Z, where Z is the number of steps. Hence, the change in each price is

$$(37) \qquad \Delta p_i = p_i(s+1) - p_i(s) = \frac{1}{Z}(p_i^T - p_i^O)$$

$$\text{where} \quad p_i(O) = p_i^O \quad \text{and} \quad p_i(Z) = p_i^T \,.$$

By choosing an arbitrarily large Z, the price change becomes infinitesimally small. This reduces the approximation error and, thus, RESORT approximates the true COLI to an arbitrary degree of accuracy, i.e., the bias of the approximation can be made infinitesimally small.

Let v be an auxiliary variable in the interval $s \le v \le s$+1. Hence, given (37), the analogous equation to (36) is

$$(38) \qquad C(s+1) = C(s) + \sum_{i=1}^{n} \int_s^{s+1} q_i^h(p(v), U^O) \, dp_i(v) \,.$$

In (38), the starting value of compensated income is C^O and the terminal value is obtained by adding to C^O the sum of the changes in compensated income from each step. That is, the solution for C^T is the value of $C(s)$ at the last step Z or

$$(39) \qquad C^T = C^O + \sum_{s=O}^{Z} (C(s+1) - C(s)) \quad \text{where} \quad C(O) = C^O \,.$$

Next, we express (38) as a Taylor series expansion around the value of compensated income, $C(s)$. For an rth-order Taylor series expansion with a remainder R,

$$(40) \qquad C(s+1) \ = \ C(s) \ + \ \sum_{m=1}^{r} \frac{1}{m!} d^{\,m} C(p(v), U^{O}) \ + \ R \, ,$$

where $d^{\,m}C(p(v), U^{O})$ is the total differential of order m of the expenditure function. The approximation achieves arbitrary accuracy depending on the highest order of the Taylor series.

Consider approximations up to the second-order. By duality,

$$(41) \qquad \frac{\partial C(p(v), U^{O})}{\partial p_i}\Big|_{v\,=\,s} \ = \ q_i^{\,h}(p(s), U^{O}) \ = \ q_i(p(s), C(s)) \, ,$$

where $q_i(p(s),C(s))$ is the ordinary demand function. That is, when compensated income is substituted into the ordinary demand functions, the quantities obtained are the compensated quantities. Moreover, the Slutsky equation is

$$(42) \qquad \frac{\partial^2 C(p,U)}{\partial p_i \partial p_j} \ = \ \frac{\partial q_i^{\,h}}{\partial p_j} \ = \ \frac{\partial q_i}{\partial p_j} \ + \ q_j \frac{\partial q_i}{\partial C} \ = \ S_{ij} \, ,$$

where S_{ij} denotes the compensated price effect.

Substituting (37), (41) and (42) into (40) and ignoring the remainder term R, the second-order Taylor series approximation $C_r(s+1)$ to the true compensated income $C(s+1)$ is

$$(43) \qquad C_r(s+1) \ = \ C_r(s) \ + \ \sum_{i=1}^{n} q_i(p(s), C_r(s))\, \Delta p_i \ + \ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} S_{ij}\, (p(s), C_r(s))\; \Delta p_i \, \Delta p_j \, .$$

The computation at $s = O$ begins with compensated income $C_r(O) = C(O) = C^{O}$ at the initial price vector $p(O) = p^{O}$. At any step s+1, the computation requires that the ordinary demand functions and their derivatives be evaluated given the known prices and the compensated income from the preceding step s. In this view, (43) is a "forward" second-order approximation.

Suppose that (43) has been computed all the way to the last step $s = Z$, where $C_r(Z)$ is the approximation to the compensated income C^{T} at the terminal price vector $p(Z) = p^{T}$. Technically, the "forward" approximation may be reversed starting with $C_r(Z)$ and $p(Z)$. That is, $C_r(s)$ is to be solved knowing $C_r(s+1)$ as prices change from $p(s+1)$ to $p(s)$. Hence, using (37), the reverse of (43) or the "backward" second-order approximation to $C(s)$ is

$$(44)\qquad C_r(s) \;=\; C_r(s+1) \;-\; \sum_{i=1}^{n} q_i(p(s+1), C_r(s+1))\,\Delta p_i$$
$$+\;\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} S_{ij}\,(p(s+1), C_r(s+1))\;\Delta p_i \Delta p_j\,.$$

The "solution" to $C_r(s)$ in (44) will not necessarily be the same as its "known" value in (43). Similarly, the "solution" to $C_r(s+1)$ in (43) will not necessarily be the same as its "known" value in (44).[26] To insure that (43) and (44) give the same values of $C_r(s)$ and $C_r(s+1)$, combine the two equations and solve $C_r(s+1)$ as the mutual unknown from

$$(45)\qquad C_r(s+1) \;=\; C_r(s) \;+\; \frac{1}{2}\sum_{i=1}^{n} q_i(p(s), C_r(s))\,\Delta p_i \;+\; \frac{1}{2}\sum_{i=1}^{n} q_i(p(s+1), C_r(s+1))\,\Delta p_i$$
$$+\;\frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} S_{ij}\,(p(s), C_r(s))\;\Delta p_i \Delta p_j$$
$$-\;\frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} S_{ij}\,(p(s+1), C_r(s+1))\;\Delta p_i \Delta p_j\,,$$

starting from $C_r(O) = C(O) = C^O$. Because $C_r(s+1)$ is in both sides of (45), the solution requires iteration. In (45), the values of $C_r(s)$ and $C_r(s+1)$ must satisfy both the "forward" solution in (43) and the "backward" solution in (44). Thus, (45) is a REversible Second-ORder Taylor (RESORT) algorithm that yields unique values of compensated income for each price vector by solving (43) and (44) simultaneously.

The reversibility property of the algorithm is important because a cost-of-living index, being simply a ratio, should change only if the base point is changed, if nothing else changes. Given this property, if the prices are reversed between any two sets of prices, the ratio of compensated incomes in one direction is the reciprocal of the ratio of compensated incomes in the reverse direction. Since this ratio is, by definition, the RESORT cost-of-living index, then RESORT satisfies Fisher's test of the "time reversal" property of ideal index numbers (Allen, 1975). The important difference, however, is that RESORT satisfies reversibility while holding the utility level constant, but the Fisher index, in contrast, satisfies reversibility without necessarily holding utility constant. Thus, the Fisher index is not necessarily a constant-utility price index—for example, under non-homothetic preferences—and this makes it a "less than ideal" index for adjusting incomes to maintain a base period standard of living.

Expenditure minimization implies that the expenditure function is concave and linearly homogeneous in prices. By concavity, the quadratic form of the second-order terms in (45) is non-positive (Varian, 1992) or

[26]While the analytic basis may not be obvious, this claim can be verified numerically.

$$(46) \qquad \sum_{i=1}^{n}\sum_{j=1}^{n} S_{ij}\,\Delta p_i\,\Delta p_j \;=\; \begin{bmatrix}\Delta p_1 & \Delta p_2 & \dots & \Delta p_n\end{bmatrix} \begin{bmatrix} S_{11} & S_{12} & \dots & S_{1n} \\ S_{21} & S_{22} & \dots & S_{2n} \\ \vdots & \dots & \ddots & \vdots \\ S_{n1} & S_{n2} & \dots & S_{nn} \end{bmatrix} \begin{bmatrix}\Delta p_1 \\ \Delta p_2 \\ \vdots \\ \Delta p_n\end{bmatrix} \le 0 .$$

The square matrix of compensated price effects, S_{ij}, is symmetric by Young's theorem because it is the Hessian of the expenditure function. The condition in (46) implies that this matrix is negative semi-definite because of concavity, or equivalently that it has non-positive eigenvalues.[27] This is necessary and sufficient for expenditure minimization or for the integrability of the demand functions.

In practical applications when only the ordinary demand functions are known, the RESORT algorithm computes S_{ij} through the Slutsky equation in (42). This equation can be rewritten in terms of expenditure shares and ordinary price and expenditure or income elasticities as

$$(47) \qquad S_{ij} \;=\; \frac{w_i\,C}{p_i\,p_j}\left[\frac{\partial q_i}{\partial p_j}\,\frac{p_j}{q_i} \;+\; \frac{\partial q_i}{\partial C}\,\frac{C}{q_i}\,w_j\right],$$

where $q_i = q_i(p, C)$ is the ordinary demand function. Zero-degree homogeneity in prices of the compensated demand function implies that[28]

$$(48) \qquad \sum_{j=1}^{n} S_{ij}\,p_j \;=\; \frac{w_i\,C}{p_i}\left[\sum_{j=1}^{n}\frac{\partial q_i}{\partial p_j}\,\frac{p_j}{q_i} \;+\; \frac{\partial q_i}{\partial C}\,\frac{C}{q_i}\sum_{j=1}^{n} w_j\right] \;=\; 0 .$$

This is true because, inside the bracketed expression, the sum of expenditure shares equals one and the sum of ordinary price and income elasticities equals zero, resulting from the property of zero-degree homogeneity in prices and income of the ordinary demand function.

Checking that both symmetry and negative semi-definiteness of the matrix in (46) are satisfied at every price set $p_i(s)$ insures that the change in compensated income corresponds to a move from an expenditure-minimizing point. This feature is important in empirical work when it cannot be presumed that the demand system is globally well-behaved, or locally well-behaved over the price range under examination.

In a two-good case where each good satisfies (48) and where the compensated cross-price effects are symmetric, a necessary and sufficient condition for (46) is that the compensated own-price effect for either good is non-positive. This implies the condition from (46) that

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[27]Linear homogeneity in prices of the expenditure function implies that this matrix is singular so that at least one eigenvalue is zero. The presence of a positive eigenvalue is sufficient evidence of a violation of concavity in prices and the absence of a zero eigenvalue is sufficient evidence of a violation of linear homogeneity.

[28]The result in (48) follows from Euler's theorem on homogeneous functions because a compensated demand function is, by Shephard's lemma, the own-price derivative of the expenditure function.

$$(49) \qquad S_{ii} = \frac{w_i\, C}{p_i^2}\left[\frac{\partial q_i}{\partial p_i}\frac{p_i}{q_i} + \frac{\partial q_i}{\partial C}\frac{C}{q_i}w_i\right] \leq 0 \quad for \quad i = 1, 2\,.$$

Applying RESORT to the AIDS Model and to the Generalized Logit Demand System

Requiring that demand functions have an explicit utility function is very limiting in practice. More importantly, this requirement is unnecessary in principle. It is necessary and sufficient that the demand system has a symmetric and negative semi-definite matrix of compensated price effects. If so, the demand system is "integrable," i.e., a utility function exists that could rationalize the demand system, although the utility function may not be recoverable.

An example of a demand system that has no explicit underlying utility function is the generalized logit model of expenditures (Dumagan & Mount, 1996; Rothman, Hong & Mount, 1994). However, the properties implied by utility maximization or expenditure minimization are embodied into the specification, as described below.

Since expenditure shares must sum to unity, let w_i follow a generalized logit specification,

$$(50) \qquad w_i = \frac{p_i\, q_i(p, C)}{C} = \frac{e^{f_i}}{e^{f_1} + \ldots + e^{f_n}} = \frac{e^{f_i}}{\sum_{i=1}^{n} e^{f_i}} \quad where \quad \sum_{i=1}^{n} w_i = 1\,.$$

This specification guarantees that each expenditure share lies between zero and one, unlike in the translog and AIDS models.[29] From (50), the ordinary demand function $q_i(p, C)$ is

$$(51) \qquad q_i(p, C) = \frac{C}{p_i}\frac{e^{f_i}}{\sum_{i=1}^{n} e^{f_i}}\,.$$

The properties of the demand system comprising $q_i(p, C)$ for n goods are embodied in the functional form of f_i. The specification below was used by Rothman, Hong, and Mount (1994):

$$(52) \qquad f_i = \alpha_i + \sum_{k=1}^{n} \delta_{ik}\,\theta_{ik} \ln\left(\frac{p_k}{p_i}\right) + \beta_i \ln\left(\frac{C}{SPI}\right) \quad where \quad \delta_{ik} = \delta_{ki} \quad for \quad i \neq k$$

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[29]Dumagan & Mount (1996) showed that the generalized logit model is, in theory, less restrictive compared to the more well-known translog and AIDS models. Moreover, Rothman, Hong & Mount (1994) showed in an empirical application to the same data set that the generalized logit violated the theoretical restrictions (e.g., negative semi-definiteness of the substitution matrix) in fewer cases compared to the more frequent violations by the translog and by the AIDS. In particular, the translog and AIDS predicted negative expenditure shares, which are not possible in the generalized logit model.

and where α, β, and δ are parameters. Let *SPI* be a Stone price index,

$$(53) \qquad \ln SPI = \sum_{i=1}^{n} w_i \ln p_i \,.$$

The cross-price weights θ_{ik} are defined by

$$(54) \qquad \theta_{ik} = w_i^{\gamma-1} w_k^{\gamma} \quad and \quad w_i \theta_{ik} = w_k \theta_{ki} \,,$$

where γ is a parameter. These weights are built into the model, together with the symmetry restrictions ($\delta_{ik} = \delta_{ki}$), to insure the symmetry of the compensated price effects. In (53) and (54), the expenditure shares are taken as "fixed" when the elasticities are derived.

From (51) and (52), the demand elasticities of the above generalized logit model are:[30]

$$(55) \qquad E_{ii} = \frac{\partial q_i}{\partial p_i}\frac{p_i}{q_i} = -\sum_{k=1}^{n} \delta_{ik}\theta_{ik} - w_i\left(\beta_i - \sum_{j=1}^{n} w_j \beta_j\right) - 1 \,,$$

$$(56) \qquad E_{ik} = \frac{\partial q_i}{\partial p_k}\frac{p_k}{q_i} = \delta_{ik}\theta_{ik} - w_k\left(\beta_i - \sum_{j=1}^{n} w_j \beta_j\right) \,,$$

$$(57) \qquad E_{iC} = \frac{\partial q_i}{\partial C}\frac{C}{q_i} = \beta_i - \sum_{j=1}^{n} w_j \beta_j + 1 \,.$$

It can be verified that

$$(58) \qquad E_{ii} + \sum_{k=1}^{n} E_{ik} + E_{iC} = 0 \quad for \quad i \neq k \,,$$

which implies that the ordinary demand functions satisfy zero-degree homogeneity in prices and income. Substituting the elasticities in (55) through (57) into the Slutsky equation in (47),

$$(59) \qquad S_{ik} = \frac{w_i C}{p_i p_k}\left(\delta_{ik}\theta_{ik} + w_k\right) \quad and \quad S_{ik} = S_{ki} \,.$$

That is, the generalized logit satisfies the symmetry of the compensated cross-price effects, given the symmetry of the price parameters in (52) and that of the cross-price weights in (54).

Consider the case of two goods. Given that zero-degree homogeneity in (58) and symmetry in (59) are satisfied, the generalized logit model is well-behaved if and only if the

[30] The price and income elasticities in (55), (56), and (57) are for the "short-run" when expenditure shares may be taken as "fixed." However, in calculating the results in Table 5 the expenditure shares are allowed to change from one price step to the next.

compensated own-price effect from (59) is non-positive for any of the two goods. That is, for good 1,

$$(60) \qquad S_{11} = - \frac{w_1 C}{p_1^2}\left(\delta_{12}\,\theta_{12} + 1 - w_1\right) \leq 0 .$$

The generalized logit model guarantees in (50) that each expenditure share lies strictly between 0 and 1. Hence, the cross-price weight (θ_{ik}) is positive. Therefore, since income or expenditure and prices are positive, the sign of (60) depends only on the sign of δ_{12}. A positive δ_{12} implies that (60) is strictly negative for all prices, so that the generalized logit is "integrable," implying that an underlying utility function exists in principle, although it is unknown.

The results in Table 5 are obtained from our earlier paper where the price data are

$$(61) \qquad \left\{p_1^0, p_2^0\right\} = \{1, 2\} \quad , \quad \left\{p_1^T, p_2^T\right\} = \{1.2375, 1.2692\} .$$

Therefore, using (37) and letting Z = 19, the prices at each step are obtained from

$$(62) \qquad p_1(s+1) = p_1(s) + \frac{1}{Z}(p_1^T - p_1^O) = p_1(s) + 0.0125 ,$$

$$(63) \qquad p_2(s+1) = p_2(s) + \frac{1}{Z}(p_2^T - p_2^O) = p_2(s) - 0.0384625 .$$

The parameters of the generalized logit model are:

$$(64) \qquad \alpha_1 = -2.119190 \quad , \quad \beta_1 = 0.5 \quad , \quad \alpha_2 = 0 \ ,$$

$$\beta_2 = 0 \quad , \quad \delta_{12} = 1.5 \quad , \quad \gamma = 2.0 \quad \textit{and} \quad C^0 = 220 .$$

Given these parameters, the total expenditure at the initial set of prices is C^O = 220. For each price set, the generalized logit is well-behaved, as shown by the negative values of the compensated own-price effect for good 1.[31] Therefore, although this model has no explicit utility function, duality implies that the initial expenditure level C^O = 220 is also the initial compensated income. Hence, we can use 220 as the initial value to compute compensated incomes and the COLI at alternative prices by applying RESORT to the generalized logit ordinary demand functions.

To see how closely RESORT approximates the true COLI, we used the ordinary demand functions of the AIDS model. However, we used a different set of parameters than those in (34). For the AIDS results in Table 5, the parameters are also from our earlier paper. These are:

[31]These are shown in Table 4 (Dumagan & Mount, 1995).

$$(65) \qquad \alpha_0 = -0.168561 \quad , \quad \beta_0 = 0.069315 \quad , \quad \alpha_1 = -0.45 \ ,$$
$$\beta_1 = 0.10 \quad , \quad \gamma_{11} = -0.20 \quad , \quad U^0 = 100 \quad and \quad C^0 = 220 \ .$$

Given the above parameters and the prices in (61), the AIDS model yields the same initial minimum expenditure as the logit model, equal to $C^O = 220$ for the utility level $U^O = 100$.[32] This utility level is held fixed to derive the true COLI of the AIDS model when prices change.

Table 5 shows that in the AIDS model the true COLI (from the expenditure function) and the RESORT approximation (from the ordinary demand functions) are equal up to three decimal places in most cases. The difference is less than 0.001 percentage point except in the last price step. This level of accuracy can be improved by increasing the number of price steps in (37), thus reducing the approximation error at each price step.

Having shown how closely RESORT approximates the true COLI in the AIDS model, using only the ordinary demand functions, we can be confident that it can approximate with the same level of accuracy the unknown COLI from ordinary demand systems with no explicit utility functions. For example, the RESORT approximation to the unknown COLI of the generalized logit model is presented in Table 5. While the prices are the same, the AIDS model yields a monotonically decreasing COLI. In contrast, the generalized logit COLI first rises up to step 6 and then starts to fall. This is simply a reflection of the difference between the structures of consumer preferences underlying the two models. However, these preferences are well-behaved because both models have negative compensated own-price effects for good 1, which are necessary and sufficient for well-behaved preferences in this two-good case.

The RESORT algorithm basically computes compensated income at alternative prices, given an initial income level. Therefore, by definition of compensated income, RESORT keeps the initial level of utility constant, although this level is unknown. Once the compensated incomes for alternative set of prices are known, the COLI is simply computed by dividing the compensated incomes by the initial income to represent the base period. Because the initial level of (unknown) utility is held fixed by the RESORT procedure, the resulting COLI is free from substitution and income biases no matter the structure of the underlying consumer preferences.

[32] Like the generalized logit model, the two-good AIDS model above is also well-behaved because the values of the compensated own-price effects for good one are negative, shown in Table 2 (Dumagan & Mount, 1995).

Table 5
RESORT Approximations to the True COLI

Price Steps	Price of Good 1	Price of Good 2	AIDS Model		Generalized Logit Model	
			True COLI (Index)	RESORT Approximation to the COLI (Index)	True COLI	RESORT Approximation to the COLI (Index)
0	1.0000	2.0000	100.0000	100.0000	Unknown	100.0000
1	1.0125	1.9615	98.5360	98.5360	Unknown	100.1774
2	1.0250	1.9231	97.0442	97.0441	Unknown	100.3243
3	1.0375	1.8846	95.5249	95.5247	Unknown	100.4401
4	1.0500	1.8461	93.9784	93.9781	Unknown	100.5241
5	1.0625	1.8077	92.4051	92.4048	Unknown	100.5757
6	1.0750	1.7692	90.8054	90.8050	Unknown	100.5939
7	1.0875	1.7308	89.1795	89.1791	Unknown	100.5780
8	1.1000	1.6923	87.5280	87.5275	Unknown	100.5271
9	1.1125	1.6538	85.8510	85.8505	Unknown	100.4404
10	1.1250	1.6154	84.1489	84.1484	Unknown	100.3167
11	1.1375	1.5769	82.4221	82.4215	Unknown	100.1552
12	1.1500	1.5384	80.6709	80.6703	Unknown	99.9546
13	1.1625	1.5000	78.8956	78.8949	Unknown	99.7138
14	1.1750	1.4615	77.0966	77.0959	Unknown	99.4316
15	1.1875	1.4231	75.2741	75.2734	Unknown	99.1065
16	1.2000	1.3846	73.4286	73.4278	Unknown	98.7372
17	1.2125	1.3461	71.5603	71.5595	Unknown	98.3221
18	1.2250	1.3077	69.6696	69.6688	Unknown	97.8596
19	1.2375	1.2692	67.7570	67.7560	Unknown	97.3479

VII. CONCLUSION

Nominal incomes do not necessarily keep pace with the true cost of living. That is, the ratio of nominal income (Y^T) in the current year to nominal income (Y^O) in the base year is not necessarily equal to the value of the true cost-of-living index (COLI) in the current year. Since the COLI is itself a ratio, with Y^O in the denominator, the difference between these ratios, (COLI - Y^T/Y^O), is the rate of income adjustment needed to maintain the base year standard of living. In practice, the "true" COLI is unknown and in its place the CPI is used. The CPI, however, is biased upward because it is based on the Laspeyres price index that has an inherently positive substitution bias. Thus, the CPI overstates the rate of income adjustment necessary to maintain the same standard of living—a problem that led the U.S. Senate Finance Committee in 1995 to appoint the Advisory Commission to Study the Consumer Price Index. The Commission recommended that the CPI should move towards a COLI concept by adopting a superlative price index, e.g., the Fisher or Tornqvist indices. Superlative indices are, however, necessarily closer to the COLI than the Laspeyres index if preferences are homothetic—an unrealistic case where all income elasticities equal 1, the true COLI is independent of the level of utility (standard of living), and expenditure shares are unaffected by changes in income. Under more realistic non-homothetic preferences, expenditure shares change with income, introducing an income bias to the Fisher and Tornqvist indices. The Laspeyres index has substitution bias, but no income bias, because it uses only fixed expenditure shares. Under plausible conditions, the combined substitution and income biases of either the Fisher or the Tornqvist index could be *negative or positive* and could be *larger* than the substitution bias of the Laspeyres index. Hence, a change in the CPI to a superlative index formula could result in under or over compensation beyond the current CPI indexation. Therefore, the Commission's recommendation to adopt a superlative index formula for the CPI needs more careful examination.

We propose RESORT as a theoretically rigorous and practical procedure to compute the COLI from ordinary demand functions, without positing a specific structure of consumer preferences. RESORT yields a COLI that is free from both substitution and income biases. In the current CPI framework, the demand systems approach to the COLI is feasible but limited to the upper level of index aggregation. Implementing the demand systems approach may impose short-term transitional problems in the current framework of timely production of the CPI for official purposes. We propose it as part of current BLS research initiatives to understand better the biases of the CPI and to move the CPI towards a more accurate measure of the COLI.

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